

# ON YAU RIGIDITY THEOREM FOR MINIMAL SUBMANIFOLDS IN SPHERES \*

JUAN-RU GU AND HONG-WEI XU

## Abstract

In this note, we investigate the well-known Yau rigidity theorem for minimal submanifolds in spheres. Using the parameter method of Yau and the DDVV inequality verified by Lu, Ge and Tang, we prove that if  $M$  is an  $n$ -dimensional oriented compact minimal submanifold in the unit sphere  $S^{n+p}(1)$ , and if  $K_M \geq \frac{\operatorname{sgn}(p-1)p}{2(p+1)}$ , then  $M$  is either a totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese surface in  $S^4(1)$ . Here  $\operatorname{sgn}(\cdot)$  is the standard sign function. We also extend the rigidity theorem above to the case where  $M$  is a compact submanifold with parallel mean curvature in a space form.

## 1 Introduction

It plays an important role in geometry of submanifolds to investigate rigidity of minimal submanifolds. After the pioneering rigidity theorem for closed minimal submanifolds in a sphere due to Simons [22], a series of striking rigidity results for minimal submanifolds were proved by several geometers [2, 13, 27]. Let  $M^n$  be an  $n$ -dimensional compact Riemannian manifold isometrically immersed into an  $(n+p)$ -dimensional complete and simply connected Riemannian manifold  $F^{n+p}(c)$  with constant curvature  $c$ . Denote by  $K_M$  and  $H$  the sectional curvature and mean curvature of  $M$  respectively. In 1975, Yau [27] first proved the following celebrated rigidity theorem for minimal submanifolds in spheres under sectional curvature pinching condition.

**Theorem A.** *Let  $M^n$  be an  $n$ -dimensional oriented compact minimal submanifold in  $S^{n+p}(1)$ . If  $K_M \geq \frac{p-1}{2p-1}$ , then either  $M$  is the totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese surface in  $S^4(1)$ .*

The pinching constant above is the best possible in the case where  $p = 1$ , or  $n = 2$  and  $p = 2$ . It's better than the pinching constant of Simons [22] in the sense of the average of sectional curvatures. Later, Itoh [12] proved that if  $M^n$  is an oriented compact minimal submanifold in  $S^{n+p}(1)$  whose sectional curvature satisfies  $K_M \geq \frac{n}{2(n+1)}$ , then  $M$  is the totally geodesic sphere or the Veronese submanifold. Further discussions in this direction

---

\*2010 Mathematics Subject Classification. 53C24; 53C40; 53C42.

Keywords: Minimal submanifold, Yau rigidity theorem, sectional curvature, mean curvature.

Research supported by the NSFC, Grant No. 11071211, 10771187; the Trans-Century Training Programme Foundation for Talents by the Ministry of Education of China.

have been carried out by many other authors (see [5, 10, 14, 21, 24, 25, 26]). An important problem is stated as follows.

**Open Problem B.** *What is the best pinching constant for the rigidity theorem for oriented compact minimal submanifolds in a unit sphere under sectional (Ricci, scalar, resp.) curvature pinching condition?*

Up to now, the problem above is still open. In particular, Lu's conjecture [18], a scalar curvature pinching problem for minimal submanifolds in a unit sphere, has not been verified yet. In this note, using Yau's parameter method [27] and the DDVV conjecture proved by Lu, Ge and Tang [7, 16], we prove the following rigidity theorem for minimal submanifolds in spheres.

**Theorem 1.** *Let  $M^n$  be an  $n$ -dimensional oriented compact minimal submanifold in the unit sphere  $S^{n+p}(1)$ . If*

$$K_M \geq \frac{\operatorname{sgn}(p-1)p}{2(p+1)},$$

*then  $M$  is either a totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese surface in  $S^4(1)$ . Here  $\operatorname{sgn}(\cdot)$  is the standard sign function.*

**Remark 1.** When  $2 < p < n$ , our pinching constant in Theorem 1 is better than ones given by Yau [27] and Itoh [12].

More generally, we obtain the following rigidity result for submanifolds with parallel mean curvature in spaces forms.

**Theorem 2.** *Let  $M^n$  be an  $n$ -dimensional oriented compact submanifold with parallel mean curvature ( $H \neq 0$ ) in  $F^{n+p}(c)$ . If  $c + H^2 > 0$  and*

$$K_M \geq \frac{\operatorname{sgn}(p-2)(p-1)}{2p}(c + H^2),$$

*then  $M$  is either a totally umbilical sphere  $S^n(\frac{1}{\sqrt{c+H^2}})$  in  $F^{n+p}(c)$ , the standard immersion of the product of two spheres or the Veronese surface in  $S^4(\frac{1}{\sqrt{c+H^2}})$ .*

## 2 Notation and lemmas

Throughout this paper, let  $M^n$  be an  $n$ -dimensional compact Riemannian manifold isometrically immersed into an  $(n+p)$ -dimensional complete and simply connected space form  $F^{n+p}(c)$  of constant curvature  $c$ . We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+p; \quad 1 \leq i, j, k, \dots \leq n; \quad n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p.$$

Choose a local field of orthonormal frames  $\{e_A\}$  in  $F^{n+p}(c)$  such that, restricted to  $M$ , the  $e_i$ 's are tangent to  $M$ . Let  $\{\omega_A\}$  and  $\{\omega_{AB}\}$  be the dual frame field and the connection 1-forms of  $F^{n+p}(c)$  respectively. Restricting these forms to  $M$ , we have

$$\begin{aligned}\omega_{\alpha i} &= \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \\ h &= \sum_{\alpha, i, j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{\alpha, i} h_{ii}^\alpha e_\alpha, \\ R_{ijkl} &= c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),\end{aligned}\tag{1}$$

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta),\tag{2}$$

where  $h, \xi, R_{ijkl}, R_{\alpha\beta kl}$ , and  $\bar{R}_{ABCD}$  are the second fundamental form, the mean curvature vector, the curvature tensor, the normal curvature tensor of  $M$ , and the curvature tensor of  $N$ , respectively. We define

$$S = |h|^2, \quad H = |\xi|, \quad H_\alpha = (h_{ij}^\alpha)_{n \times n}.$$

The scalar curvature  $R$  of  $M$  is given by

$$R = n(n-1)c + n^2 H^2 - S.$$

Denote  $K_M(p, \pi)$  the sectional curvature of  $M$  for tangent 2-plane  $\pi \subset T_p M$  at point  $p \in M$ . Set  $K_{\min}(p) = \min_{\pi \subset T_p M} K_M(p, \pi)$ . From [27], we have the following lemma.

**Lemma 1.** *If  $M^n$  is a submanifold with parallel mean curvature and positive sectional curvature in  $F^{n+p}(c)$ , then  $M$  is a pseudo-umbilical submanifold.*

Let  $M$  be a submanifold with parallel mean curvature vector  $\xi$ . Choose  $e_{n+1}$  such that it is parallel to  $\xi$ , and

$$\text{tr} H_{n+1} = nH, \quad \text{tr} H_\alpha = 0, \quad \alpha \neq n+1.\tag{3}$$

Set

$$S_H = \text{tr} H_{n+1}^2, \quad S_I = \sum_{\alpha \neq n+1} \text{tr} H_\alpha^2.\tag{4}$$

When  $M$  is a pseudo-umbilical submanifold, we have

$$S_H = \text{tr} H_{n+1}^2 = nH^2.\tag{5}$$

Denoting the first and second covariant derivatives of  $h_{ij}^\alpha$  by  $h_{ijk}^\alpha$  and  $h_{ijkl}^\alpha$  respectively, we have

$$\begin{aligned}\sum_k h_{ijk}^\alpha \omega_k &= dh_{ij}^\alpha - \sum_k h_{ik}^\alpha \omega_{kj} - \sum_k h_{kj}^\alpha \omega_{ki} - \sum_\beta h_{ij}^\beta \omega_{\beta\alpha}, \\ \sum_l h_{ijkl}^\alpha \omega_l &= dh_{ijk}^\alpha - \sum_l h_{ijl}^\alpha \omega_{lk} - \sum_l h_{ilk}^\alpha \omega_{lj} - \sum_l h_{ljk}^\alpha \omega_{li} - \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}.\end{aligned}$$

Then we have

$$h_{ijk}^\alpha = h_{ikj}^\alpha, \quad h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{mj}^\alpha R_{mikl} - \sum_\beta h_{ij}^\beta R_{\alpha\beta kl},$$

$$\Delta h_{ij}^\alpha = \sum_k h_{ijk}^\alpha = \sum_k h_{kij}^\alpha + \sum_k \left( \sum_m h_{km}^\alpha R_{mijk} + \sum_m h_{mi}^\alpha R_{mkjk} - \sum_\beta h_{ki}^\beta R_{\alpha\beta jk} \right). \quad (6)$$

The following lemma will be used in the proof of our main results.

**Lemma 2**([27]). *If  $M^n$  is a submanifold with parallel mean curvature in  $F^{n+p}(c)$ , then either  $H \equiv 0$  or  $H$  is non-zero constant and  $H_{n+1}H_\alpha = H_\alpha H_{n+1}$  for all  $\alpha$ .*

The DDVV inequality proved by Lu, Ge and Tang [7, 16] is stated as follows.

**DDVV Inequality.** *Let  $B_1, \dots, B_m$  be symmetric  $(n \times n)$ -matrices, then*

$$\sum_{r,s=1}^m \|[B_r, B_s]\|^2 \leq \left( \sum_{r=1}^m \|B_r\|^2 \right)^2, \quad (7)$$

where the equality holds if and only if under some rotation all  $B_r$ 's are zero except two matrices which can be written as

$$\tilde{B}_r = P \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t, \quad \tilde{B}_s = P \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t,$$

where  $P$  is an orthogonal  $(n \times n)$ -matrix. Here  $\|\cdot\|^2$  denotes the sum of squares of entries of the matrix and  $[A, B] = AB - BA$  is the commutator of the matrices  $A, B$ .

For further discussions about the DDVV inequality, we refer to see [4, 7, 15, 16, 17, 18].

### 3 Proof of the theorems

When  $M^n$  be a minimal submanifold in  $S^{n+p}(1)$ , we have  $\text{tr} H_\alpha = 0$  for all  $\alpha$  and  $\sum_i h_{iikl}^\alpha = 0$ . It follows from (6) that

$$\Delta h_{ij}^\alpha = \sum_{k,m} h_{km}^\alpha R_{mijk} + \sum_{k,m} h_{mi}^\alpha R_{mkjk} - \sum_{k,\beta} h_{ki}^\beta R_{\alpha\beta jk}. \quad (8)$$

Thus

$$\sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{i,j,k,m,\alpha} (h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}) - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk}. \quad (9)$$

**Proof of Theorem 1.** By using (1) and (2), we get

$$\begin{aligned} & \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \\ &= nS + \sum_{\alpha,\beta} \text{tr} H_\beta \cdot \text{tr}(H_\alpha^2 H_\beta) - \sum_{\alpha,\beta} [\text{tr}(H_\alpha H_\beta)]^2 - \sum_{\alpha,\beta} [\text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2], \end{aligned}$$

and

$$\sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} = \sum_{\alpha,\beta} [\text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2].$$

Since  $(\text{tr}(H_\alpha H_\beta))$  is a symmetric  $(p \times p)$ -matrix, we can choose the normal frame fields  $\{e_\alpha\}$  such that

$$\text{tr}(H_\alpha H_\beta) = \text{tr} H_\alpha^2 \cdot \delta_{\alpha\beta}.$$

This implies

$$\sum_{\alpha,\beta} [\text{tr}(H_\alpha H_\beta)]^2 = \sum_{\alpha} (\text{tr} H_\alpha^2)^2. \quad (10)$$

From above equalities, we obtain

$$\begin{aligned} \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha &= -anS + (1+a) \sum_{i,j,k,m,\alpha} (h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}) \\ &\quad + (a-1) \sum_{\alpha,\beta} [\text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2] + a \sum_{\alpha,\beta} (\text{tr} H_\alpha^2)^2, \end{aligned} \quad (11)$$

for all real number  $a$ . For fixed  $\alpha$ , we choose the orthonormal frame fields  $\{e_i\}$  such that  $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$ . Hence, we get

$$\begin{aligned} & \sum_{i,j,k,m} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \\ &= \sum_{i,k} \lambda_i^\alpha \lambda_k^\alpha R_{k i i k} + \sum_{i,k} \lambda_i^\alpha \lambda_i^\alpha R_{i k i k} \\ &= \frac{1}{2} \sum_{i,j} (\lambda_i^\alpha - \lambda_j^\alpha)^2 R_{ijij} \\ &\geq \frac{1}{2} K_{\min} \sum_{i,j} (\lambda_i^\alpha - \lambda_j^\alpha)^2 \\ &= nK_{\min} (\text{tr} H_\alpha^2), \end{aligned} \quad (12)$$

which implies that

$$\sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \geq nK_{\min} S. \quad (13)$$

On the other hand, by a direct computation and the DDVV inequality, we obtain

$$\begin{aligned} \sum_{\alpha,\beta} \text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2 &= \frac{1}{2} \sum_{\alpha,\beta} \text{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\ &\leq \frac{1}{2} \text{sgn}(p-1) \left( \sum_{\alpha} \text{tr} H_\alpha^2 \right)^2 \\ &= \frac{1}{2} \text{sgn}(p-1) S^2, \end{aligned} \quad (14)$$

where  $\text{sgn}(\cdot)$  is the standard sign function. It follows from (11), (13) and (14) that

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &\geq \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - anS + (1+a)nK_{\min}S + \left[ \frac{a}{p} + \frac{\text{sgn}(p-1)}{2}(a-1) \right] S^2, \end{aligned} \quad (15)$$

for  $0 \leq a < 1$ . Taking  $a = \text{sgn}(p-1)\frac{p}{p+2}$ , we get

$$\frac{1}{2}\Delta S \geq nS \left[ \left( 1 + \text{sgn}(p-1)\frac{p}{p+2} \right) K_{\min} - \text{sgn}(p-1)\frac{p}{p+2} \right].$$

It follows from the assumption and the maximum principal that  $S$  is a constant, and

$$S \left[ \left( 1 + \text{sgn}(p-1)\frac{p}{p+2} \right) K_{\min} - \text{sgn}(p-1)\frac{p}{p+2} \right] = 0.$$

If there is a point  $q \in M$  such that  $K_{\min}(q) > \frac{\text{sgn}(p-1)p}{2(p+1)}$ , then  $S = 0$ , i.e.,  $M$  is totally geodesic. If  $K_{\min} \equiv \frac{\text{sgn}(p-1)p}{2(p+1)}$ , then inequalities in (13), (14) and (15) become equalities. From the DDVV inequality we obtain  $p \leq 2$ . This together with Theorem A implies  $M$  is the product of two spheres or the Veronese surface in  $S^4(1)$ . This completes the proof of Theorem 1.

When  $M^n$  is a submanifold with parallel mean curvature in  $F^{n+p}(c)$ , we have  $\xi = He_{n+1}$ , and  $\sum_i h_{iikl}^\alpha = 0$  for  $\alpha \neq n+1$ . It follows from (6) and Lemma 2 that

$$\Delta h_{ij}^\alpha = \sum_{k,m} h_{km}^\alpha R_{mijk} + \sum_{k,m} h_{mi}^\alpha R_{mkjk} - \sum_{k,\beta \neq n+1} h_{ki}^\beta R_{\alpha\beta jk}, \quad \alpha \neq n+1. \quad (16)$$

Thus

$$\begin{aligned} \sum_{i,j,\alpha \neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha &= \sum_{i,j,k,m,\alpha \neq n+1} (h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}) \\ &\quad - \sum_{i,j,k,\alpha,\beta \neq n+1} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk}. \end{aligned} \quad (17)$$

**Proof of Theorem 2.** Applying (1) and (2), we get

$$\begin{aligned} &\sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \\ &= ncS_I + \sum_{\alpha \neq n+1, \beta} \text{tr} H_\beta \cdot \text{tr}(H_\alpha^2 H_\beta) - \sum_{\alpha \neq n+1, \beta} [\text{tr}(H_\alpha H_\beta)]^2 \\ &\quad - \sum_{\alpha, \beta \neq n+1} [\text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2], \end{aligned}$$

and

$$\sum_{i,j,k,\alpha,\beta \neq n+1} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} = \sum_{\alpha,\beta \neq n+1} [\text{tr}(H_\alpha^2 H_\beta^2) - \text{tr}(H_\alpha H_\beta)^2].$$

Since  $\alpha, \beta \neq n+1$ ,  $(tr(H_\alpha H_\beta))$  is a symmetric  $(p-1) \times (p-1)$ -matrix. We choose the normal vector fields  $\{e_\alpha\}_{\alpha \neq n+1}$  such that

$$tr(H_\alpha H_\beta) = tr H_\alpha^2 \cdot \delta_{\alpha\beta},$$

which implies

$$\sum_{\alpha, \beta \neq n+1} [tr(H_\alpha H_\beta)]^2 = \sum_{\alpha \neq n+1} tr(H_\alpha^2)^2. \quad (18)$$

For any real number  $a$ , we have

$$\begin{aligned} \sum_{i,j,\alpha \neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha &= (1+a) \sum_{i,j,k,m,\alpha \neq n+1} (h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}) - anc S_I \\ &\quad + (a-1) \sum_{\alpha, \beta \neq n+1} [tr(H_\alpha^2 H_\beta^2) - tr(H_\alpha H_\beta)^2] + a \sum_{\alpha \neq n+1} (tr H_\alpha^2)^2 \\ &\quad + a \left\{ - \sum_{\alpha \neq n+1} tr(H_\alpha^2 H_{n+1}) \cdot tr H_{n+1} + \sum_{\alpha \neq n+1} [tr(H_\alpha H_{n+1})]^2 \right\}. \end{aligned} \quad (19)$$

When  $p = 1$ ,  $M$  is a compact hypersurface with nonzero constant mean curvature and nonnegative sectional curvature in  $F^{n+1}(c)$ . The assertion was proved by Nomizu and Smyth [19] for  $c \geq 0$  and by Walter [23] for  $c < 0$ , respectively.

When  $p = 2$ ,  $K_M \geq 0$  and  $H = \text{constant} \neq 0$ . We know from Theorem 9 in [27] that  $M$  is a minimal hypersurface in the totally umbilical sphere  $S^{n+1}\left(\frac{1}{\sqrt{c+H^2}}\right)$ . This together with Theorem A implies that  $M$  is either a totally umbilical sphere or the standard immersion of the product of two spheres.

When  $p \geq 3$ , it follows from Lemma 1 and the assumption that  $M$  is pseudo-umbilical, i.e.,  $h_{ij}^{n+1} = H\delta_{ij}$ . Hence, we have

$$\begin{aligned} &\sum_{\alpha \neq n+1} tr(H_\alpha^2 H_{n+1}) \cdot tr H_{n+1} - \sum_{\alpha \neq n+1} [tr(H_\alpha H_{n+1})]^2 \\ &= \sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1} h_{kk}^{n+1} - \sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{km}^\alpha h_{mk}^{n+1} h_{ij}^{n+1} \\ &= nH^2 \sum_{i,j,\alpha \neq n+1} (h_{ij}^\alpha)^2 - H^2 \sum_{\alpha \neq n+1} (tr H_\alpha)^2 \\ &= nH^2 S_I. \end{aligned} \quad (20)$$

On the other hand, we get from (12)

$$\sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \geq nK_{\min} S_I. \quad (21)$$

By a direct computation and the DDVV inequality, we obtain

$$\begin{aligned} \sum_{\alpha, \beta \neq n+1} tr(H_\alpha^2 H_\beta^2) - tr(H_\alpha H_\beta)^2 &= \frac{1}{2} \sum_{\alpha, \beta \neq n+1} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\ &\leq \frac{1}{2} \left( \sum_{\alpha \neq n+1} tr H_\alpha^2 \right)^2 \\ &= \frac{1}{2} S_I^2. \end{aligned} \quad (22)$$

It follows from (19), (20), (21) and (22) that

$$\begin{aligned}
\frac{1}{2}\Delta S_I &= \sum_{i,j,\alpha \neq n+1} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha \neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha \\
&\geq (1+a)nK_{\min}S_I + a \sum_{\alpha \neq n+1} (tr H_\alpha^2)^2 + \frac{1}{2}(a-1)S_I^2 - an(c+H^2)S_I \\
&\geq (1+a)nK_{\min}S_I + \left(\frac{a}{p-1} + \frac{a-1}{2}\right)S_I^2 - an(c+H^2)S_I \\
&= S_I \left[ (1+a)nK_{\min} + \left(\frac{a}{p-1} + \frac{a-1}{2}\right)S_I - an(c+H^2) \right], \tag{23}
\end{aligned}$$

for  $0 \leq a < 1$ . Taking  $a = \frac{p-1}{p+1}$ , we get

$$\begin{aligned}
\frac{1}{2}\Delta S_I &\geq nS_I[(1+a)K_{\min} - a(c+H^2)] \\
&= nS_I \left[ \left(1 + \frac{p-1}{p+1}\right)K_{\min} - \frac{p-1}{p+1}(c+H^2) \right].
\end{aligned}$$

It follows from the assumption and the maximum principal that  $S_I$  is a constant, and

$$S_I \left[ \left(1 + \frac{p-1}{p+1}\right)K_{\min} - \frac{p-1}{p+1}(c+H^2) \right] = 0.$$

If there is a point  $q \in M$  such that  $K_{\min}(q) > \frac{(p-1)(c+H^2)}{2p}$ , then  $S_I = 0$ , i.e.,  $M$  is a compact hypersurface with nonzero constant mean curvature and positive sectional curvature in a totally geodesic submanifold  $F^{n+1}(c)$ . Therefore,  $M$  is a totally umbilical sphere  $S^n(\frac{1}{\sqrt{c+H^2}})$ .

If  $K_{\min} \equiv \frac{(p-1)(c+H^2)}{2p}$ , then inequalities in (21), (22) and (23) become equalities. This together with the DDVV inequality implies that  $p = 3$  and  $K_{\min} = \frac{c+H^2}{3}$ . Taking  $a = 0$  in (23), we get  $S_I = \frac{2n}{3}(c+H^2)$ . By the same argument as in [2], we conclude that  $n = 2$ . Hence,  $K_M = \frac{c+H^2}{3}$  and  $M$  is the Veronese surface in  $S^4(\frac{1}{\sqrt{c+H^2}})$ . This completes the proof of Theorem 2.

Combing Theorems 1, 2 and rigidity results in [12, 21, 26], we present a general version of the Yau rigidity theorem.

**Generalized Yau Rigidity Theorem.** *Let  $M^n$  be an  $n$ -dimensional oriented compact submanifold with parallel mean curvature in  $F^{n+p}(c)$ , where  $c + H^2 > 0$ . Set  $k(m, n) = \min\{\text{sgn}(m-1)m, n\}$ . Then we have*

(i) if  $H = 0$  and

$$K_M \geq \frac{k(p, n)c}{2[k(p, n) + 1]},$$

then  $M$  is either a totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese submanifold;



(ii) if  $H \neq 0$  and

$$K_M \geq \frac{k(p-1, n)(c + H^2)}{2[k(p-1, n) + 1]},$$

then  $M$  is either a totally umbilical sphere  $S^n(\frac{1}{\sqrt{c+H^2}})$  in  $F^{n+p}(c)$ , the standard immersion of the product of two spheres, or the Veronese submanifold.

Recently Andrews and Baker [1] generalized a weaker version of Huisken's convergence theorem [8] for mean curvature flow of convex hypersurfaces in  $\mathbf{R}^{n+1}$  to higher codimensional cases. Motivated by Generalized Yau Rigidity Theorem, we would like to propose the following conjecture on mean curvature flow in higher codimensions, which can be considered as a generalization of the Huisken convergence theorem [8].

**Conjecture.** Let  $M_0 = F_0(M)$  be an  $n$ -dimensional compact submanifold in an  $(n+p)$ -dimensional space form  $F^{n+p}(c)$  with  $c + H^2 > 0$ . If the sectional curvature of  $M_0$  satisfies

$$K_M > \frac{k(p, n)(c + H^2)}{2[k(p, n) + 1]},$$

then the mean curvature flow

$$\begin{cases} \frac{\partial}{\partial t} F(x, t) = n\xi(x, t), & x \in M, t \geq 0, \\ F(\cdot, 0) = F_0(\cdot). \end{cases} \quad (24)$$

has a unique smooth solution  $F : M \times [0, T) \rightarrow F^{n+p}(c)$  on a finite maximal time interval, and  $F_t(\cdot)$  converges uniformly to a round point  $q \in F^{n+p}(c)$  as  $t \rightarrow T$ .

When  $p = 1$  and  $c = 0$ , the conjecture was verified by Huisken [8]. When  $p = 1$  and  $c = 1$ , a weaker version of the conjecture was proved by Huisken [9].

## References

- [1] B. Andrews and C. Baker, Mean curvature flow of pinched submanifolds to spheres, *J. Differential Geom.*, **85**(2010), 357-396.
- [2] S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, in *Functional Analysis and Related Fields*, Springer-Verlag, New York(1970).
- [3] T. Choi and Z. Lu, On the DDVV conjecture and the comass in calibrated geometry (I), *Math. Z.*, **260**(2008), 409-429.
- [4] P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, A pointwise inequality in submanifold theory, *Arch. Math.*, **35**(1999), 115-128.
- [5] N. Ejiri, Compact minimal submanifolds of a sphere with positive Ricci curvature, *J. Math. Soc. Japan.*, **31**(1979), 251-256.

- [6] J. Erbacher, Reduction of the codimension of an isometric immersion. *J. Differential Geom.*, **5**(1971), 333-340.
- [7] J. Q. Ge and Z. Z. Tang, A proof of the DDVV conjecture and its equality case, *Pacific J. Math.*, **237**(2008), 87-95.
- [8] G. Huisken, Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.*, **20**(1984), 237C266.
- [9] G. Huisken, Deforming hypersurfaces of the sphere by their mean curvature, *Math. Z.*, **195**(1987), 205C219.
- [10] M. Kozłowski and U. Simon, Minimal immersions of 2-manifolds into spheres, *Math. Z.*, **186**(1984), 377-382.
- [11] T. Itoh, On veronese manifolds, *J. Math. Soc. Japan.*, **27**(1975), 497-506.
- [12] T. Itoh, Addendum to my paper "On veronese manifolds", *J. Math. Soc. Japan.*, **30**(1978), 73-74.
- [13] B. Lawson, Local rigidity theorems for minimal hyperfaces, *Ann. of Math.*, **89**(1969), 187-197.
- [14] A. M. Li and J. M. Li, An intrinsic rigidity theorem for minimal submanifolds in a sphere, *Arch. Math.*, **58**(1992), 582-594.
- [15] Z. Lu, On the DDVV conjecture and the comass in calibrated geometry (II), arXiv:math.DG/0708.2921v1.
- [16] Z. Lu, Proof of the normal scalar curvature conjecture, arXiv:math.DG/0711.3510v1.
- [17] Z. Lu, Recent developments of the DDVV conjecture, *Bull. Transil. Univ. Brasov serB.*, **14**(2008), 133-144.
- [18] Z. Lu, Normal scalar curvature conjecture and its applications, arXiv:math.DG/0803.0502.
- [19] K. Nomizu and B. Smyth, A formula of Simons' type and hypersurfaces with constant mean curvature, *J. Differential Geom.*, **3**(1969), 367-377.
- [20] K. Shiohama and H. W. Xu, A general rigidity theorem for complete submanifolds, *Nagoya Math. J.*, **150**(1998), 105-134.
- [21] Y. B. Shen, Submanifolds with nonnegative sectional curvature, *Chinese Ann. Math. SerB.*, **5**(1984), 625-632.
- [22] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. Math.*, **88**(1986), 62-105.
- [23] R. Walter, Compact hypersurfaces with a constant higher mean curvature function, *Math. Ann.*, **270**(1985), 125-145.
- [24] H. W. Xu, A rigidity theorem for submanifolds with parallel mean curvature in a sphere, *Arch. Math.*, **61**(1993), 489-496.

- [25] H. W. Xu, On closed minimal submanifolds in pinched Riemannian manifolds, *Trans. Amer. Math. Soc.*, **347**(1995), 1743-1751.
- [26] H. W. Xu and W. Han, Geometric rigidity theorem for submanifolds with positive curvature, *Appl. Math. J. Chinese Univ. Ser. B*, **20**(2005), 475-482.
- [27] S. T. Yau, Submanifolds with constant mean curvature I, II, *Amer. J. Math.*, **96**, **97**(1974, 1975), 346-366, 76-100.

Juan-Ru Gu  
 Center of Mathematical Sciences  
 Zhejiang University  
 Hangzhou 310027  
 China  
 E-mail address: gujr@cms.zju.edu.cn

Hong-Wei Xu  
 Center of Mathematical Sciences  
 Zhejiang University  
 Hangzhou 310027  
 China  
 E-mail address: xuhw@cms.zju.edu.cn